The Univalence Maxim and Univalent Double Categories

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Sameness and Isomorphism

- Often mathematical structures are **considered up to isomorphism** rather than up to equality
- Examples: groups, rings, R-modules, vector spaces, ...
- **Isomorphism implies "sameness"**: isomorphic groups have the same group-theoretic properties, isomorphic rings have the same ring-theoretic properties, and so on
- Isomorphic objects are identified

Sameness and Isomorphism in Univalent Foundations

In Univalent Foundations, we have the following theorems:

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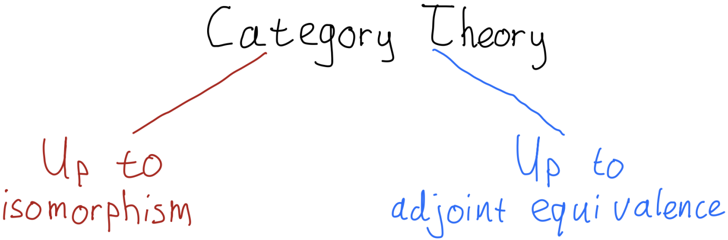
Let G and H be groups. Then the map sending identites $G = H$ to isomorphisms $G \cong H$ is an equivalence of types.

Concretely, this means:

- Two sets have the same properties if they are isomorphic
- Two groups have the same properties if they are isomorphic

Sameness for Categories



Sameness for Categories

Category Theory

Up to isomorphism

(cannot be disregarded too much)

Up to adjoint equivalence

Our favorite!

Categories in Univalent Foundations

Definition

A category is called a **setcategory** if its type of objects is a set.

Definition

A category is called **univalent** if the map from identities $x = y$ to isomorphisms $x \cong y$ is an equivalence of types.

Univalence Principles for Categories

In Univalent Foundations, we have the following theorems:

Theorem (Ahrens, Kapulkin, Shulman, 2015)

Let $\mathcal{C}$ and $\mathcal{D}$ be setcategories. Then the map sending identites $\mathcal{C} = \mathcal{D}$ to **isomorphisms** *$\mathcal{C} \cong \mathcal{D}$ is an equivalence of types.*

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Concretely, this means:

- Two setcategories have the same properties if they are isomorphic
- Two univalent categories have the same properties if they are equivalent

But what about higher categories?

Higher categories come with

- more notions of equivalence
- more notions of strictness

For instance, we have the following notions of equivalence

- for **2-categories and bicategories**: isomorphism, essentially surjective & local isomorphism, biequivalence
- for **double categories**: isomorphism, vertical equivalence, gregarious equivalence

Our Philosophy

We introduce the **univalence maxim**:

For each notion of equivalence of a given categorical structure, there exists a tailored definition whose notion of equality in univalent foundation precisely coincides with the chosen notion of equivalence

This talk: apply the univalence maxim to double categories

What Are Double Categories?

A double category is given by

- objects
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- vertical morphisms
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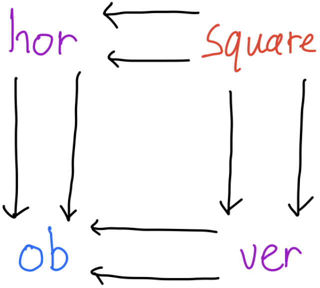
Note:

- We have **identity and composition operations** for vertical and horizontal morphisms, and for squares
- Composition for morphisms could either be **strictly** unital and associative or **weakly**
- So: double categories come with **various notions of strictness**

What Are Double Categories?



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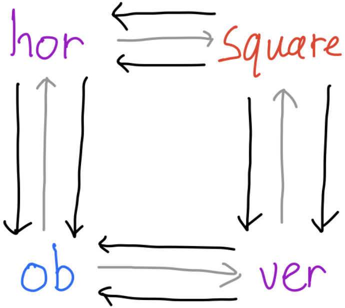

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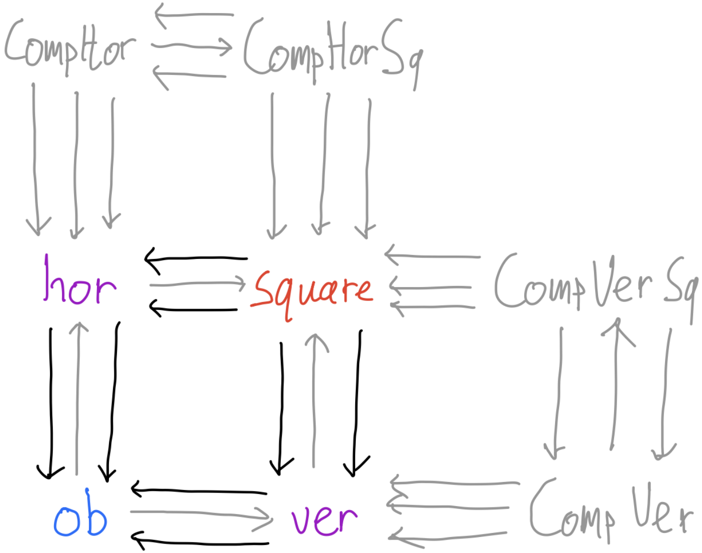

Flavors of Double Categories

There are different flavors of double categories:

- Strict double categories: strict in both directions
- Pseudo double categories: weak in 1 direction
- Weak double categories: weak in both directions

Examples of Double Categories

	Objects	Horizontal	Vertical	Kind
Rel	sets	functions	relations	strict
Span($\mathcal{C}$)	$x, y : \mathcal{C}$	$x \to y$	spans in $\mathcal{C}$	pseudo
$\mathsf{Prof}_{\mathsf{set}}$	setcategories	functors	profunctors	pseudo
$\mathsf{Prof}_{\mathsf{univ}}$	univalent cats	functors	profunctors	weak
Sq($\mathcal{B}$)	objects in $\mathcal{B}$	1-cells	1-cells	weak

Here $\mathcal{C}$ is a category with pullbacks and $\mathcal{B}$ is a bicategory

Sameness of Double Categories

Double Categories

- Up to isomorphism — strict in 2 directions
- Up to vertical equivalence — strict in 1 direction
- Up to gregarious equivalence — strict in 0 directions

A Zoo of Double Categories

We acquired various notions of double categories

- Strict double setcategories (up to isomorphism)
- Pseudo double setcategories (up to isomorphism)
- Weak double setcategories (up to isomorphism)
- Univalent pseudo double categories (up to vertical equivalence)
- Univalent weak double categories (up to gregarious equivalence)

Conclusion

- Higher categories come with various notions of equivalence
- **Univalence maxim**: for every flavor of equivalence of structured categories, find a suitable notion whose identity corresponds to those equivalences
- We applied this to double categories
- **Question**: general framework for the univalence maxim

Relevant literature:

- Univalent Double Categories[1]
- Insights From Univalent Foundations: A Case Study Using Double Categories[2]

[1] https://dl.acm.org/doi/10.1145/3636501.3636955
[2] https://arxiv.org/abs/2402.05265